Algebraic Geometry of the Center-Focus problem for Abel Differential Equation

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Abstract

The Abel differential equation $y' = p(x)y^3 + q(x)y^2$ with polynomial coefficients p, q is said to have a center on $[a, b]$ if all its solutions, with the initial value $y(a)$ small enough, satisfy the condition $y(a) = y(b)$. The problem of giving conditions on (p, q, a, b) implying a center for the Abel equation is analogous to the classical Poincaré Center-Focus problem for plane vector fields.

Center conditions are provided by an infinite system of "Center Equations". An important new information on these equations has been obtained via a detailed analysis of two related structures: Composition Algebra and Moment Equations (first order approximation of the Center ones). Recently one of the basic open questions in this direction - the "Polynomial moments problem" - has been completely settled in [16, 17, 18].

In this paper we present a progress in the following two main directions: First, we translate the results of [16, 17, 18] into the language of Algebraic Geometry of the Center Equations. On this base we obtain new information on the center conditions, significantly extending, in particular, the results of [3]. Second, we study the "second Melnikov coefficients" (second order approximation of the Center equations) showing that in many cases vanishing of the moments and of these coefficients is sufficient in order to completely characterize centers.

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1 Introduction

We consider the Abel differential equation

$$y' = p(x)y^3 + q(x)y^2 \tag{1.1}$$

with meromorphic coefficients p, q. A solution $y(x)$ of (1.1) is called "closed" along a curve γ with the endpoints a, b if $y(a) = y(b)$ for the initial element of $y(x)$ around a analytically continued to b along γ. Equation (1.1) is said to have a center along γ if any its solution $y(x)$ with the initial value $y(a)$ small enough, is closed along γ. We always will assume that γ is a (closed or non-closed) curve avoiding singularities of p and q. Furthermore, in this paper we always will assume that p, q are polynomials, and will denote by P, Q the primitives $P(x) = \int_a^x p(\tau)d\tau$ and $Q(x) = \int_a^x q(\tau)d\tau$.

The Center-Focus problem for the Abel equation is to give a necessary and sufficient condition on p, q, γ for (1.1) to have a center along γ. The Smale-Pugh problem is to bound the number of isolated closed solutions of (1.1). The relation of these problems to the classical Hilbert 16-th and Poincaré Center-Focus problems for plane vector fields is well known (see, eg. [5, 11]).

Algebraic Geometry enters the above problems from the very beginning: it is well known that center conditions are given by an infinite system of polynomial equations in the coefficients of p, q, expressed as certain iterated integrals of p, q ("Center Equations"; see Section 3 below). The structure of the ideal generated by these equations in an appropriate ring (Bautin ideal) determines local bifurcations of the closed solutions as p, q vary.

One of the main difficulties in the Center-Focus and the Smale-Pugh problems is that a general algebraic-geometric analysis of the system of Center Equations is very difficult because of their complexity and absence of apparent general patterns.

In recent years the following two important algebraic-analytic structures, deeply related to the Center Equations for (1.1), have been discovered: Composition Algebra of polynomials and generalized polynomial moments of the form $m_k = \int_\gamma P^k(x)q(x)dx$ (the last as a special case of iterated integrals). The use of these structures provides important tools for investigation of the Center-Focus problem for the Abel equation (see [1, 2, 3, 4, 7, 5, 9, 13] and references therein). In particular, it was shown in [3] that Center Equations are closely approximated by the Moment Equations $m_k = \int_\gamma P^k(x)q(x)dx = 0$,

and in fact coincide with them "at infinity". Moment Equations, in turn, impose (in many cases) strong restrictions on P and Q, considered as elements of the Composition Algebra. (Notice that usually *linear* Moment Equations $m_k = 0$ are considered, where P is fixed while Q is the unknown. However, consideration of Center Equations at infinity in [3] and in the present paper leads to a *non-linear* setting where Q is fixed, while the equations have to be solved with respect to the unknown P).

The following Composition Condition imposed on P and Q plays a central role in the study of the Moment and Center Equations (see the references above): there exist polynomials $\widetilde{P}$, $\widetilde{Q}$ and W with $W(a) = W(b)$ such that

$$P(x) = \widetilde{P}(W(x)),\ Q(x) = \widetilde{Q}(W(x)).$$

Being a kind of "integrability condition", Composition Condition implies vanishing of Center and Moments Equations as well as of all the iterated integrals entering the Center Equations. It is the only known to us *sufficient* center condition for the polynomial Abel equation. Using the interrelation between Center and Moment Equations at infinity, and the Composition Condition, a rather accurate description of the affine Center Set for the Abel equation has been given in [3]. Very recently important results relating Center and Composition Conditions for trigonometric and polynomial Abel equations have been obtained in [6, 7, 9].

In [17, 18] a serious progress has been achieved in understanding both the Moments Equations and the relevant Composition Algebra. In particular, explicit necessary and sufficient conditions for vanishing of all m_k have been given there.

One of the main goals of the present paper is to give an algebraic-geometric interpretation of the results of [16, 12, 17, 18] in the context of the Center-Focus problem for the polynomial Abel equation. Here we work with the Moment Equations, which form the "first order approximation" of the Center Equations. On this base we obtain, following the approach of [3], new information on the center conditions, significantly extending the results of [3].

The second main goal is to start the investigation of the "second Melnikov coefficients", which form the second order approximation of the Center Equations. We show that in many important cases vanishing of the moments and of the Melnikov coefficients is sufficient in order to completely characterize centers.

A general form of the results in this paper is the following: we show that the Composition Condition is indeed a very good approximation to the Center Conditions. In various circumstances we provide an upper bound for the dimension of the non-composition components of the Center Set. In many cases this bound is zero, so the Center Set coincides with the Composition Set up to a finite number of points. The following theorems summarize our main new results on the center configurations for polynomial Abel equation (1.1). Since there is a one-to-one correspondence between pairs of polynomials p, q and pairs of their primitives P, Q defined above, we will formulate all our results in terms of P and Q.

Theorem 1.1 *Consider equation (1.1) with Q fixed and P varying in the space $\mathcal{P}_d$ of all the polynomials of the degree up to d vanishing at a and b. Then the dimension of the non-composition components of the Center Set of (1.1) does not exceed $[\frac{d}{6}]+2$.*

In many cases the general bound provided by Theorem 1.1 may be improved. In order to formulate corresponding results it is convenient to normalize points a and b to be the points $-\frac{\sqrt{3}}{2}$ and $\frac{\sqrt{3}}{2}$, respectively. Further, let $\mathcal{S} \subset \mathcal{P}$ be a subset of all polynomials $Q \in \mathcal{P}$ representable as a sum $Q = S_1(T_2)+S_2(T_3)$, where S_1, S_2 are arbitrary polynomials, while T_2, T_3 are the Chebyshev polynomials of the degree 2 and 3, respectively (notice that the normalization is chosen in such a way that $T_2(a) = T_2(b)$, $T_3(a) = T_3(b)$). Below we show that the dimension of $\mathcal{S} \cap \mathcal{P}_d$ does not exceed $[\frac{2}{3}d]+1$, so "most" of the polynomials Q of degree d cannot be represented in the above form.

Theorem 1.2 *Let P vary in the space $\mathcal{P}_9$. Then for each fixed $Q \in \mathcal{P} \setminus \mathcal{S}$ the Center Set of (1.1) consists of a Composition Set and possibly a finite set of additional points. For an arbitrary fixed Q the dimension of the non-composition components of the Center Set of (1.1) does not exceed one. For P varying in the space $\mathcal{P}_{11}$ and for an arbitrary fixed Q the dimension of the non-composition components of the Center Set of (1.1) does not exceed two.*

The next result heavy relies on computations with the second Melnikov coefficients.

Theorem 1.3 *Let P vary in the space $\mathcal{P}_9$. Then for each fixed $Q \in \mathcal{S} \cap \mathcal{P}_8$, such that nether T_2 nor T_3 is a right composition factors of Q, the Center Set*

of (1.1) consists of a Composition Set and possibly a finite set of additional points.

Our last result concerns the Center Set in subspaces of polynomials with a special structure. Let U_d be the space consisting of all polynomials $P \in \mathcal{P}_d$ such that the degrees of x, appearing in P with the non-zero coefficients, are powers of prime numbers.

Theorem 1.4 *Let P vary in U_d. Then for any fixed Q the Center Set of (1.1) in U_d consists of a Composition Set and possibly a finite set of additional points.*

2 Preliminaries: Poincaré mapping, Center Equations, and Composition condition

2.1 Poincaré mapping and Center Equations

Both the Center-Focus and the Smale-Pugh problems can be naturally expressed in terms of the Poincaré "first return" mapping $y_b = G_\gamma(y_a)$ along γ. Let $y(x, y_a)$ denote the element around a of the solution $y(x)$ of (1.1) satisfying $y(a) = y_a$. The Poincaré mapping G_γ associates to each initial value y_a at a the value y_b at b of the solution $y(x, y_a)$ analytically continued along γ.

According to the definition above, the solution $y(x, y_a)$ is closed along γ if and only if $G_\gamma(y_a) = y_a$. Therefore closed solutions correspond to the fixed points of G_γ, and (1.1) has a center if and only if $G_\gamma(y) \equiv y$. It is well known that $G_\gamma(y)$ for small y is given by a convergent power series

$$G_\gamma(y) = y + \sum_{k=2}^{\infty} v_k(p, q, \gamma) y^k. \tag{2.1}$$

Therefore the center condition $G_\gamma(y) \equiv y$ is equivalent to an infinite sequence of algebraic equations on p and q:

$$v_k(p, q, \gamma) = 0, \ \ k = 2, 3, \ldots. \tag{2.2}$$

Each $v_k(p, q, \gamma)$ can be expressed as a linear combination of certain iterated integrals of p and q along γ (see, for example, [3]). For the purposes of the

present paper we do not need the exact form of these equations but only some results about their behavior "at infinity" presented below.

2.2 Projective setting and Center Equations at infinity

Let $\mathcal{P} = \mathcal{P}_{[a,b]}$ be the vector space of all complex polynomials P satisfying $P(a) = P(b) = 0$, and $\mathcal{P}_d$ the subspace of $\mathcal{P}$ consisting of polynomials of degree at most d. We always shall assume that the polynomials

$$P(x) = \int_a^x p(\tau)d\tau, \quad Q(x) = \int_a^x q(\tau)d, \tag{2.3}$$

defined above are elements of $\mathcal{P}$. This restriction is natural in the study of the center conditions since it is forced by the first two of the Center Equations. Since (2.3) provides a one to one correspondence between (p, q) and (P, Q), in order to avoid a cumbersome notation all the results below are formulated in terms of (P, Q).

In the polynomial setting of the Center-Focus problem the choice of the integration path γ from a to b is not essential. Indeed, we assume that y_a is small, and hence movable singularities of the solutions are "far away". So we may put $\gamma = [a, b]$. We shall assume that the points $a \neq b$ are fixed and usually shall omit a, b from the notation.

From now on we shall assume that $Q \in \mathcal{P}_{d_1}$ is fixed, while P varies in a certain linear subspace V of the space $\mathcal{P}_d$. This restrictive setting significantly simplifies the presentation. See [3] for a comparison of this and other possible settings.

Let a subspace $V \subset \mathcal{P}$ be given. We shall consider the projective space PV and the infinite hyperplane $HV \subset PV$. To construct PV we introduce an auxiliary variable $\nu \in \mathbb{C}$ and consider the couples (S, ν) with (S, ν) and $(\lambda S, \lambda\nu)$ identified for $\lambda \neq 0$.

Let us denote by $\hat{v}_k(p, q)$ the "homogenization" of the Center Equations $v_k = 0$ with respect to the variable P. In other words, we multiply each term in v_k by an appropriate degree of an auxiliary variable ν to make v_k homogeneous.

We call "Center Equations at infinity" the restrictions of the homogeneous Center Equations to the infinite hyperplane HV. They are obtained by putting $\nu = 0$ in the homogeneous equations described above.

Theorem 2.1 *([3]) For $k = 2, 4, \ldots$ even, and $l = \frac{k}{2} - 1$ the Center Equations at infinity are given by the generalized moments*

$$v_k^\infty(P, Q) = m_l(P, Q) = \int_a^b P^l(x) q(x) dx = 0. \tag{2.4}$$

For k odd the Center Equations at infinity are given by the coefficients of the "second Melnikov function"

$$v_k^\infty(P, Q) = D_k(P, Q) = 0, \tag{2.5}$$

represented by certain linear combinations of iterated integrals in p, q with exactly two appearances of q.

2.3 Center, Moment, and Composition Sets

Let us assume that $Q \in \mathcal{P}_{d_1}$ and a subspace $V \subset \mathcal{P}_d$ are fixed. We define the Center Set $CS = CS_{V,Q}$ as the set of $P \in V$ for which equation (1.1) has a center. Equivalently, CS is the set of $P \in V$ satisfying Center Equations (2.2). The moment set $MS = MS_{V,Q}$ consists of $P \in V$ satisfying Moment equations (2.4).

To introduce Composition Set $COS = COS_{V,Q}$ we recall the polynomial Composition Condition defined in [2], which is a special case of the general Composition Condition introduced in [1] (for brevity below we will use the abbreviation "CC" for the "Composition Condition").

Definition 2.1 *Polynomials P, Q are said to satisfy the "Composition Condition" on $[a, b]$ if there exist polynomials $\widetilde{P}$, $\widetilde{Q}$ and W with $W(a) = W(b)$ such that P and Q are representable as*

$$P(x) = \widetilde{P}(W(x)), \quad Q(x) = \widetilde{Q}(W(x)).$$

The Composition Set $COS_{V,Q}$ consists of all $P \in V$ for which P and Q satisfy the Composition Condition.

It is easy to see that the Composition Condition implies center for (1.1), as well as the vanishing of each of the moments and iterated integrals above. So we have $COS \subset CS$, $COS \subset MS$. Examples and partial results (see [3, 7] for the most recent contributions) seem to support the following "Composition conjecture":

Conjecture 1. *The Center and Composition Sets for any polynomial Abel equation coincide.*

Define $\bar{CS}, \bar{MS}, C\bar{O}S$ as the intersections of the corresponding affine sets with the infinite hyperplane HV. It follows directly from by Theorem 2.1 that the following statement is true.

Proposition 2.1 $C\bar{O}S \subset \bar{CS} \subset \bar{MS}$.

Notice that COS and MS are homogeneous, and hence these sets are cones over $\bar{MS}, C\bar{O}S$. However, CS a priori may be not homogeneous, and the connection of the affine part CS to $\bar{CS}$ may be more complicated.

Our main goal will be to compare the affine Center Set CS the Composition Set COS. For this purpose we shall bound the dimension of the affine non-composition components of CS, analyzing their possible behavior at infinity (Sections 5,6). To obtain these bounds we first describe the geometry of the Composition Set COS (Section 3) and compare the Moment set MS and its subset COS (Section 4).

3 The structure of the Composition Set

Geometry of the Composition Set reflects the algebraic structure of polynomial compositions, which is well known to provide rather subtle phenomena. In comparison with the classical theory developed by Ritt ([19]) we are interested in what we call below $[a, b]$-compositions, i.e. compositions of polynomials under requirement that some of the factors take equal values at the points a and b.

3.1 Elements of Ritt's theory

Let us recall first some basic facts on polynomial composition algebra, including the classical first and second Ritt theorems.

Definition 3.1 *A polynomial P is called indecomposable if it cannot be represented as $P(x) = R \circ S(x) = R(S(x))$ for polynomials R and S of degree greater than one. A decomposition $P = P_1 \circ P_2 \circ \cdots \circ P_r$ is called maximal if all $P_1, \ldots, P_r$ are indecomposable and of degree greater than one. Two decompositions $P = P_1 \circ P_2 \circ \cdots \circ P_r$ and $P = Q_1 \circ Q_2 \circ \cdots \circ Q_r$, maximal or not, are called equivalent (notation "$\sim$") if there exist polynomials of degree one*

$\mu_i,\ i=1,\dots,r-1$, *such that* $P_1 = Q_1\circ\mu_1,\ P_i = \mu_{i-1}^{-1}\circ Q_i\circ\mu_i,\ i=2,\dots,r-1,$ *and* $P_r = \mu_{r-1}^{-1}\circ Q_r$.

The first Ritt theorem states that any two maximal decompositions of a polynomial P have an equal number of terms, and can be obtained from one another by a sequence of transformations replacing two successive terms $A\circ C$ with $B\circ D$, such that

$$A\circ C = B\circ D. \tag{3.1}$$

Let us mention that decompositions of a polynomial P into a composition of two polynomials, up to equivalence, correspond in a one-to-one way to imprimitivity systems of the monodoromy group G_P of P. In their turn imprimitivity systems of G_P are in a one-to-one correspondence with subgroups A of G containing the stabilizer G_ω of a point $\omega\in G$. In particular, for a given polynomial P the number of its right composition factors W, up to the change $W\to\lambda\circ W$, where λ is a polynomial of degree one, is finite. Below we shall call (with a slight abuse of notation) two right composition factors W and $\lambda\circ W$ of P, where λ is a polynomial of degree one, equivalent, and write $W\sim\lambda\circ W$. We also usually shall write just "right factor" of P instead of "compositional right factor".

The first Ritt theorem reduces the description of maximal decompositions of polynomials to the description of indecomposable polynomial solutions of the equation (3.1). It is convenient to start with the following result ([8]): if polynomials A,B,C,D satisfy (3.1), then there exist polynomials $U,V,\hat{A},\hat{B},\hat{C},\hat{D}$, where

$$\deg U = \gcd(\deg A,\deg B),\quad \deg V = \gcd(\deg C,\deg D), \tag{3.2}$$

such that

$$A = U\circ\hat{A},\quad B = U\circ\hat{B},\quad C=\hat{C}\circ V,\quad D=\hat{D}\circ V, \tag{3.3}$$

and

$$\hat{A}\circ\hat{C} = \hat{B}\circ\hat{D}. \tag{3.4}$$

In particular, if $\deg A=\deg B$, then necessarily $A\circ C$ and $B\circ D$ are equivalent as decompositions. More generally, if $\deg B|\deg A$, then there exists a polynomial W such that the equalities

$$A = B\circ W,\quad D = W\circ C$$

are satisfied.

Note that the above result is equivalent to the statement that the lattice of imrimitivity systems of the monodromy group G of a polynomial P of degree n is isomorphic to a sublattice of the lattice L_n consisting of all divisors of n, where by definition

$$d_1 \wedge d_2 = \mathrm{GCD}(d_1, d_2), \quad d_1 \vee d_2 = \mathrm{LCM(d_1, d_2)}$$

(see [12]). For example, for the polynomials z^n the corresponding lattices consist of all divisors of n since for any $d|n$ the equality $z^n = z^d \circ z^{n/d}$ holds. The same is true for the Chebyshev polynomials T_n since the equality $T_n(\cos\phi) = \cos n\phi$ implies that $T_n = T_d \circ T_{n/d}$ for any $d|n$. On the other hand, for an indecomposable polynomial P the corresponding lattice contains only elements 1 and n.

The second Ritt theorem states that if A, B, C, D satisfy (3.1) and degrees of A and B as well as of C and D are coprime, then there exist linear polynomials U, V such that (3.3) and (3.4) hold, and, up to a possible replacement of $\hat{A}$ by $\hat{B}$ and $\hat{C}$ by $\hat{D}$, either

$$\hat{A} \circ \hat{C} \sim z^n \circ z^r R(z^n), \qquad \hat{B} \circ \hat{D} \sim z^r R^n(z) \circ z^n, \tag{3.5}$$

where $R(z)$ is a polynomial, $r \geq 0, n \geq 1$, and $\mathrm{GCD}(n, r) = 1$, or

$$\hat{A} \circ \hat{C} \sim T_n \circ T_m, \;\; \hat{B} \circ \hat{D} \sim T_m \circ T_n, \tag{3.6}$$

where T_n and T_m are the Chebyshev polynomials, $n, m \geq 1$, $\mathrm{GCD}(n, m) = 1$. In particular, this holds when A, B, C, D solving (3.1) are indecomposable, and the decompositions $A \circ C$ and $B \circ D$ are non-equivalent, since in this case the degrees of polynomials U, V in (3.2) and (3.3) are necessarily equal to one.

Clearly, the second Ritt theorem together with the previous result imply the following statement: if A, B, C, D satisfy (3.1), then there exist polynomials U, V such that (3.2), (3.3), (3.4) hold, and, up to a possible replacement of $\hat{A}$ by $\hat{B}$ and $\hat{C}$ by $\hat{D}$, either (3.5) or (3.6) holds.

3.2 $[a, b]$-Compositions

Now we return to $[a, b]$-compositions, i.e. compositions of polynomials under the requirement that some of the factors take equal values at two distinct points a and b.

Definition 3.2 *Let a polynomial P satisfying $P(a) = P(b)$ be given. We call polynomial W a right $[a, b]$-factor of P if $P = \widetilde{P} \circ W$ for some polynomial $\widetilde{P}$, and $W(a) = W(b)$. A polynomial P is called $[a, b]$-indecomposable, if $P(a) = P(b)$ and P does not have right $[a, b]$-factors non-equivalent to P itself.*

Remark. Notice that any right $[a, b]$-factor of P necessary has degree greater than one, and that $[a, b]$-indecomposable P may be decomposable in the usual sense.

Proposition 3.1 *Any polynomial P up to equivalence has a finite number of $[a, b]$-indecomposable right factors $W_j, \quad j = 1, \ldots, s$. Furthermore, each right $[a, b]$-factor W of P can be represented as $W = \widetilde{W}(W_j)$ for some polynomial $\widetilde{W}$ and $j = 1, \ldots, s$.*

Proof: As it was mentioned above up to equivalence there are only finitely many general right factors W of P. In particular, this is true for $[a, b]$-indecomposable right $[a, b]$-factors W_j of P.

Now let W be a right $[a, b]$-factor of P. If it is $[a, b]$-indecomposable, then by the first part of the proposition $W = \lambda \circ W_j$ for some $j = 1, \ldots, s$. Otherwise, W can be represented as $W = V \circ \hat{W}$, where $\hat{W}$ is a right $[a, b]$-factor of P and $\deg V > 1$. Since $\deg \hat{W} < \deg W$, it is clear that continuing this process we ultimately will find an $[a, b]$-indecomposable right factor W_j of P such that $W = \widetilde{W}(W_j)$. $\square$

An easy consequence of Proposition 3.1 is the following description of the Composition Set given in [3]:

Proposition 3.2 *Let $W_j, \ j = 1, \ldots, s$, be all indecomposable right $[a, b]$-factors of Q. Then the set $COS_{V,Q}$ is a union of the linear subspaces $L_j \subset V$, $j = 1, \ldots, s$, where L_j consists of all the polynomials $P \in V$ representable as $P = \widetilde{P}(W_j), \quad j = 1, \ldots, s$, for a certain polynomial $\widetilde{P}$.*

It has been recently shown in [18] that for any $P \in \mathcal{P}$ the number s of its non-equivalent $[a, b]$-indecomposable right factors can be at most three. Moreover, if $s > 1$ then these factors necessarily have a very special form, similar to what appears in Ritt's description above.

The precise statement is given by the following theorem ([18], Theorem 5.3):

Theorem 3.1 *Let complex numbers $a \neq b$ be given. Then for any polynomial $P \in \mathcal{P}_{[a,b]}$ the number s of its $[a,b]$-indecomposable right factors W_j, up to equivalence, does not exceed 3.*

Furthermore, if $s = 2$, then either

$$P = U \circ z^{rn} R^n(z^n) \circ U_1, \; W_1 = z^n \circ U_1, \; W_2 = z^r R(z^n) \circ U_1,$$

where R, U, U_1 are polynomials, $r > 0, n > 1$, $\mathrm{GCD}(n, r) = 1$, or

$$P = U \circ T_{nm} \circ U_1, \; W_1 = T_n \circ U_1, \; W_2 = T_m \circ U_1,$$

where U, U_1 are polynomials, $n, m > 1$, $\gcd(n, m) = 1$.

On the other hand, if $s = 3$ then

$$P = U \circ z^2 R^2(z^2) \circ T_{m_1 m_2} \circ U_1,$$

$$W_1 = T_{2m_1} \circ U_1, \quad W_2 = T_{2m_2} \circ U_1, \quad W_3 = zR(z^2) \circ T_{m_1 m_2} \circ U_1,$$

where R, U, U_1 are polynomials, $m_1, m_2 > 1$ are odd, and $\mathrm{GCD}(m_1, m_2) = 1$.

Notice that in all the cases above $U_1(a) \neq U_1(b)$ while $W_j(a) = W_j(b)$.

We are interested in the stratification of the space $\mathcal{P}_d$ of polynomials P of degree d according to the structure of their $[a,b]$-indecomposable right $[a,b]$-factors. Following Theorem 3.1 let us use the following notation for the appropriate strata:

Definition 3.3 *Let $DEC_s^d(a,b) \subset \mathcal{P}_d$ denote the set of polynomials P of degree at most d satisfying $P(a) = P(b) = 0$ and possessing exactly s non-equivalent $[a,b]$-indecomposable right factors. For $s = 1$ we write $DEC_1^d(a,b) = DEC_{1,0}^d(a,b) \cup DEC_{1,1}^d(a,b)$ where for $P \in DEC_{1,0}^d(a,b)$ ($P \in DEC_{1,1}^d(a,b)$) the corresponding factor is equivalent (not equivalent) to P.*

As a first consequence of Theorem 3.1 we get upper bounds on the dimensions of the sets $DEC_s^d(a,b)$ considered as subsets of the algebraic variety $\mathbb{C}^{d-1}$ identified with $\mathcal{P}_d$.

Proposition 3.3 *$DEC_{1,0}^d(a,b)$ consists of $[a,b]$-indecomposable polynomials $P \in \mathcal{P}_d$, and its dimension is $d-1$. We have $DEC_{1,1}^d(a,b) = \emptyset$ for $d \leq 3$, and $\dim DEC_{1,1}^d(a,b) \leq [\frac{d}{2}]$ for $d \geq 4$. $DEC_2^d(a,b) = \emptyset$ for $d \leq 5$, and $\dim DEC_2^d(a,b) \leq [\frac{d}{6}] + 1$ for $d \geq 6$. $DEC_3^d(a,b) = \emptyset$ for $d \leq 89$, and $\dim DEC_3^d(a,b) \leq [\frac{d}{90}]$ for $d \geq 90$.*

Proof: Assume we are given l parametric families of polynomials $\mathcal{S}_r = \{S_r(\tau_r, z)\},\ r = 1, ..., l$, with $\tau_r \in T_r \subset \mathbb{C}^{n_r}$ being the parameters of $\mathcal{S}_r$. We assume that the degree of the polynomials $S_r(\tau_r, z)$ remains constant and equal to d_r for all the values of the parameters $\tau_r \in T_r$. Put $\tau = (\tau_1, \ldots, \tau_l)$, and let

$$P_\tau = S_1(\tau_1) \circ S_2(\tau_2) \circ \cdots \circ S_l(\tau_l).$$

The degree of the polynomials P_τ of this form is $d_1 \cdot ... \cdot d_l$ and they form a parametric family with the parameters $\tau = (\tau_1, ...\tau_l) \in \mathbb{C}^n$, where $n = n_1 ... + n_l$.

The dimension D of the stratum S in $\mathcal{P}$ formed by the polynomials P_τ as above is at most n, and it may be strictly less than n since the parametric representation as above may be redundant. The requirement $P_\tau \in \mathcal{P}_d$ is equivalent to $d_1 \cdot ... \cdot d_l \leq d$. So to bound from above the dimension D of the stratum S we have to accurately estimate $D \leq n_1 + ... + n_l$, taking into account the redundancy in the parametric representation, and then to maximize D under the constraint $d_1 \cdot ... \cdot d_l \leq d$.

Let us now consider the sets $DEC^d_s(a, b)$ for $s = 1, 2, 3$ case by case. For $s = 1$ any $P \in DEC^d_{1,0}(a, b)$ is $[a, b]$-indecomposable, according to Definition 3.2. As we shall see below, all the other strata have dimension strictly smaller than $\dim \mathcal{P}_d = d - 1$. Hence $\dim DEC^d_{1,0}(a, b) = d - 1$. Now, each $P \in DEC^d_{1,1}(a, b)$ has a form $P = S_1 \circ S_2$, with $\deg S_1 = d_1 > 1, \deg S_2 = d_2 > 1$, since we assume that P possesses a right $[a, b]$-factor S_2, not equivalent to P. In this case $d \geq d_1 d_2$ is at least 4, and S_1 and S_2 can be any polynomials of degrees d_1 and d_2 with the only restrictions $S_2(a) = S_2(b)$ and $S_1(S_2(a)) = 0$. Hence $n_1 = d_1,\ n_2 = d_2$. On the space $\mathbb{C}^{n_1+n_2}$ of the parameters of (S_1, S_2) acts a two-dimensional group Γ of linear polynomials γ. It acts by transforming (S_1, S_2) into $(S_1 \circ \gamma, \gamma^{-1} S_2)$. This action preserves P. Accordingly, we have to maximize $D = d_1 + d_2 - 2$ under the constrain $d_1 d_2 \leq d$. For d even this maximum is achieved for $d_1 = 2$ or $d_2 = 2$ and it is $\frac{d}{2}$. For d odd still $d_1 = 2$ or $d_2 = 2$, but the maximum of D is $\frac{d-1}{2}$. Finally we get $\dim DEC^d_1(a, b) \leq [\frac{d}{2}]$.

Now let us consider the case $s = 2$. In this case by Theorem 3.1 we have two options.

The first option is that $P = U \circ z^{rn} R^n(z^n) \circ U_1$, where $U(z), R(z), U_1(z)$ are polynomials, $r > 0, n > 1$, and $\gcd(n, r) = 1$, and z^n and $z^r R(z^n)$ take equal values at $U_1(a) \neq U_1(b)$.

Here, denoting the degrees of U, U_1, R by $k, m, l \geq 1$, respectively, we get $\deg P = k \cdot n(r + ln) \cdot m \geq 6$, while the number of the independent parameters, i.e. the dimension of the corresponding strata is at most $k + l + m - 1$ (we take into account the requirements $W_1(a) = W_1(b),\ W_2(a) = W_2(b),\ P(a) = P(b) = 0$, and the fact that the scaling parameters of U and of R act equivalently on P). So we have to maximize $k + l + m - 1$ under the constrain $k \cdot n(r + ln) \cdot m \leq d$. The variables are integers $k \geq 1, l \geq 1, m \geq 1, r \geq 1, n \geq 2,\ \gcd(n, r) = 1$.

Let us first fix l, r, n. As above, the maximum of $k + l + m - 1$ is attained either for $k = 1, m = [\frac{d}{n(r+ln)}]$, or for $k = [\frac{d}{n(r+ln)}], m = 1$. In both cases it is $l + [\frac{d}{n(r+ln)}]$, and this expression increases as l decreases. So we can put $l = 1$ and so we get $[\frac{d}{n(r+n)}] + 1$. Once more, this expression increases as n, r (which do not enter the maximized sum) decrease. Their minimal possible values are $r = 1, n = 2$ and we get $k + l + m - 1 = [\frac{d}{6}] + 1$.

The second option is that $P = U \circ T_{nm} \circ U_1$, with $n, m > 1,\ \gcd(n, m) = 1$, and T_m and T_n take equal values at $U_1(a)$ and $U_1(b)$. Denote the degrees of U and U_1 by k and l, respectively, we get $\deg P = klmn \geq 6$, while the number of the independent parameters, i.e. the dimension of the corresponding strata, is at most $k + l - 1$ (we take into account the requirements that T_m and T_n take equal values at $U_1(a)$ and $U_1(b)$, and $P(a) = P(b) = 0$). By exactly the same reasoning as above we get the maximal dimension of the corresponding strata is achieved as either $\deg U = 1$ or $\deg U_1 = 1$, and it is at most $[\frac{d}{mn}]$. The minimal possible values for m, n here are 2 and 3, so we get the bound $[\frac{d}{6}]$ which is smaller than the one above.

It remains to consider the case $s = 3$. In this case by Theorem 3.1 we have $P = U \circ z^2 R^2(z^2) \circ T_{m_1 m_2} \circ U_1$, with U, R, U_1 as above, $m_1, m_2 > 1$ odd, and $\gcd(m_1, m_2) = 1$. In addition, T_{2m_1}, T_{2m_2} and $zR(z^2) \circ T_{m_1 m_2}$ take equal values at $U_1(a) \neq U_1(b)$.

As above, denoting the degrees of U, U_1, R by k, m, l, respectively, we get $\deg P = k \cdot (4l+2) m_1 m_2 \cdot m \geq 90$. The number of the independent parameters, i.e. the dimension of the corresponding strata, is here at most $k+l+m-2$ (we take into account, besides the requirements that W_1, W_2, W_3 take equal values at a, b, and $P(a) = P(b) = 0$, also the fact that the scaling parameters of U and of R act equivalently on P). Maximizing the last expression exactly as above, we conclude that the maximum is achieved for $l = 1, m_1 = 3, m_2 = 5$, and either $k = 1, m = [\frac{d}{(4l+2)m_1 m_2}] = [\frac{d}{90}]$, or $m = 1, k = [\frac{d}{90}]$. This maximum

is equal to $[\frac{d}{90}]$. This completes the proof of Proposition 3.3. $\square$.

Based on Proposition 3.3 and Theorem 3.1 we can now give a much more accurate description of the Composition Set $COS_{V,Q}$ for $V = \mathcal{P}_d$ or $V \subset \mathcal{P}_d$:

Theorem 3.2 *For any polynomial Q of degree at most 5 the Composition Set $COS_{V,Q}$ is a linear subspace in $\mathcal{P}_d$ with $\dim L \le [\frac{d}{2}]$. For $6 \le \deg Q \le 89$ the set $COS_{V,Q}$ is a union of at most two linear subspaces in $\mathcal{P}_d$, and for $\deg Q \ge 90$ the set $COS_{V,Q}$ is a union of at most three linear subspaces. The dimension of each of these subspaces is at most $[\frac{d}{2}]$, their double and triple intersections have dimensions at most $[\frac{d}{6}]+1$ and $[\frac{d}{90}]$, respectively.*

Proof: Let $W_j, \ j = 1, \dots, s$ be all the mutually prime right $[a,b]$-factors of Q. By Proposition 3.3, for Q of degree at most 5 we have $s = 1$. For $6 \le \deg Q \le 89$ we have $s \le 2$ and for $\deg Q \ge 90$ we have $s \le 3$. Next, by Proposition 3.2, $COS_{\mathcal{P}_d,Q}$ is a union of linear subspaces $L_j = \{P \in \mathcal{P}_d, \ P = \widetilde{P}(W_j)\}$.

Next notice that if $\deg W_j = d$, then L_j is one-dimensional, and if $\deg W_j < d$, then $L_j \subset DEC^d_{1,1}(a,b)$. We also have $L_i \cap L_j \subset DEC^d_2(a,b), \ L_i \cap L_j \cap L_k \subset DEC^d_3(a,b)$. All the required bounds on the dimensions of L_j now follow directly from Proposition 3.3. $\square$

Remark. In fact, the dimensions of the linear subspaces L_j and of their intersections may be strongly smaller than the bounds in Theorem 3.2. The reason is that in this theorem we do not take into account, for example, the fact, that if Q has mutually prime right $[a,b]$-factors W_1, W_2, then their degrees, by Theorem 3.1, cannot both be equal to two. Another reason is that in the setting of Theorem 3.2 the right factors are fixed, while in Proposition 3.3 they are variable, which also decreases the dimensions of the strata of $COS_{\mathcal{P}_d,Q}$ in comparison with the strata $DEC^d_s(a,b)$.

4 Moment vanishing versus Composition

The main result of [18] can be formulated as follows:

Theorem 4.1 *Let P with $P(a) = P(b)$ be given, and let W_j, $j = 1, \dots, s$, be all its non-equivalent $[a,b]$-indecomposable right $[a,b]$-factors. Then for any polynomial Q all the moments $m_k = \int_a^b P^k(x)q(x)dx, \ k \ge 0$, vanish if and only if $Q = \sum_{j=1}^s Q_j$, where $Q_j = \widetilde{Q}_j(W_j)$ for some polynomial $\widetilde{Q}_j$.*

This theorem combined with Theorem 3.1 provides an explicit description for vanishing of the polynomial moments. In order to use it for the study of the Moment Set, let us introduce the notions of "definite" and "co-definite" polynomials.

Definition 4.1 *Let $V, V_1 \subset \mathcal{P} = \mathcal{P}_{[a,b]}$ be fixed linear spaces. A polynomial $P \in \mathcal{P}$ is called V_1-definite if for any polynomial $Q \in V_1$ vanishing of the moments $m_k = \int_a^b P^k(x)q(x)dx$, $k \geq 0$, implies Composition Condition on $[a,b]$ for P and Q. The set of such P is denoted D_{V_1}.*

A polynomial $Q \in \mathcal{P}$ is called V-co-definite if for any polynomial $P \in V$ vanishing of the moments $m_k = \int_a^b P^k(x)q(x)dx$, $k \geq 0$, implies Composition Condition on $[a,b]$ for P and Q. The set of such Q is denoted COD_V.

If $V_1 = \mathcal{P}$ or $V = \mathcal{P}$ we call polynomials defined above $[a,b]$-definite or $[a,b]$-co-definite correspondingly, and denote their sets by D or COD.

Definite polynomials have been initially introduced and studied in [15]. Some their properties have been described in [16]. The notion of a co-definite polynomials is apparently new (although some examples have appeared in [3]). Below we give a characterization of definite and co-definite polynomials, but many questions still remain open.

4.1 Definite polynomials

Theorem 4.1 allows us to give a complete description of $[a,b]$-definite polynomials:

Theorem 4.2 *A polynomial P is $[a,b]$-definite if and only if it has, up to equivalence, exactly one $[a,b]$-indecomposable right factor W.*

Proof: Assume that P has exactly one $[a,b]$-indecomposable right factor W. By Theorem 4.1 for any polynomial Q vanishing of m_k for all $k \geq 0$ implies that there exist $\widetilde{Q}$ such that $Q = \widetilde{Q}(W)$, so Composition Condition on $[a,b]$ is satisfied for P and Q. Hence, by Definition 4.1, P is $[a,b]$-definite.

Assume now that P has two non-equivalent $[a,b]$-indecomposable right factors W_1, W_2, and show that the solution $Q = W_1 + W_2$ cannot be represented in the form $Q = \widetilde{Q}(W)$, where W is an $[a,b]$-right factor of P and $\widetilde{Q}$ is a polynomial (cf. [14]). First observe that W_1 and W_2 have different degrees for otherwise equalities (3.3) imply that W_1 and W_2 are equivalent.

Thus, without loss of generality we may assume that $\deg W_2 > \deg W_1$, and so $\deg Q = \deg W_2$, implying that if $Q = \widetilde{Q}(W)$ then $\deg W | \deg W_2$. Therefore, using (3.3) again, we conclude that $W_2 = U(W)$ for some polynomial U. Furthermore, if $\deg W < \deg W_2$, then we obtain a contradiction with the assumption that W_2 is an $[a, b]$-indecomposable right factor of P. On the other hand, if $\deg W = \deg W_2$, then as above we conclude that W and W_2 are linear equivalent implying that $W_1 = Q - W_2$ is a polynomial in W_2 in contradiction with the assumption $\deg W_2 > \deg W_1$. $\square$

Corollaries 4.1-4.2 below were proved in [16]. Here we give another proofs of these results basing on Theorem 4.2 and the second Ritt theorem. We believe that these "more algebraic" proofs clarify to some extent the structure of definite polynomials, which still presents a lot of open questions (see [15]). We also extend a classification of non-definite polynomials whose degree does not exceed nine, given in [16], up to degree eleven.

Corollary 4.1 *Let p be a prime. Then each polynomial P of degree p^s, $s \geq 1$, is $[a, b]$-definite for any $a, b \in \mathbb{C}$.*

Proof: Indeed, since imprimitivity systems of G_P form a sublattice of L_{p^s} (see definition on page 9), if W_1, W_2 are arbitrary right factors of P, then either W_1 is a polynomial in W_2 or W_2 is a polynomial in W_1. Therefore, such P can not have two non equivalent $[a, b]$-indecomposable right factors. $\square$

Corollary 4.2 *If at least one of points a and b is not a critical point of a polynomial P, then P is $[a, b]$-definite.*

Assume that P is not $[a, b]$-definite and let W_1, W_2 be its non linear equivalent $[a, b]$-indecomposable right factors. Then the second Ritt theorem implies that there exist polynomials of degree one μ_1, μ_2 and polynomials U, W such that either

$$P = U \circ z^{rs} R^n(z^n) \circ W, \quad W_1 = \mu_1 \circ z^n \circ W, \quad W_2 = \mu_2 \circ z^s R(z^n) \circ W, \quad (4.1)$$

where R is a polynomial and $GCD(s, n) = 1$, or

$$P = U \circ T_{nm} \circ W, \quad W_1 = \mu_1 \circ T_n \circ W, \quad W_2 = \mu_2 \circ T_m \circ W, \qquad (4.2)$$

where T_n, T_m are the Chebyshev polynomials and $GCD(n,m) = 1$. Furthermore, since W_1, W_2 are $[a,b]$-indecomposable and non equivalent, the inequality $W(a) \neq W(b)$ holds. In particular, $n > 1$, since $W_1(a) = W_1(b)$.

It is easy to see that if (4.1) holds, then the equalities

$$W_1(\widetilde{a}) = W_1(\widetilde{b}), \quad W_2(\widetilde{a}) = W_2(\widetilde{b}),$$

where

$$\widetilde{W}_1 = z^n, \quad \widetilde{W}_2 = z^s R(z^n), \quad \widetilde{a} = W(a), \quad \widetilde{b} = W(b),$$

taking into account the equality $GCD(s,n) = 1$, imply that the number $\widetilde{a}^n = \widetilde{b}^n$ is a root of the polynomial R. It follows now from the first formula in (4.1) by the chain rule that both a and b are critical points of P.

If (4.2) holds, then, taking into account the identity

$$T_l \circ \frac{1}{2}\left(z + \frac{1}{z}\right) = \frac{1}{2}\left(z + \frac{1}{z}\right) \circ z^l \tag{4.3}$$

and the eqaulity $GCD(m,n) = 1$, it is easy to see that there exist $\alpha, \beta \in \mathbb{C}$ such that

$$\widetilde{a} = \frac{1}{2}\left(\alpha + \frac{1}{\alpha}\right), \quad \widetilde{b} = \frac{1}{2}\left(\beta + \frac{1}{\beta}\right), \quad \alpha^n = \beta^n, \quad \alpha^m = \frac{1}{\beta^m}, \tag{4.4}$$

where as above $\widetilde{a} = W(a), \widetilde{b} = W(b)$. Furthermore, $\alpha^2 \neq 1$. Indeed, otherwise the equalities

$$\bar{\alpha}^n = \bar{\beta}^n, \quad \bar{\alpha}^m = \frac{1}{\bar{\beta}^m},$$

where $\bar{\alpha} = \alpha^2$, $\bar{\beta} = \beta^2$, taking into account the equality $GCD(m,n) = 1$, imply that $\beta^2 = 1$. Since $\widetilde{a} \neq \widetilde{b}$ this yields that either $\widetilde{a} = -1$, $\widetilde{b} = 1$, or $\widetilde{a} = 1$, $\widetilde{b} = -1$. On the other hand, since $GCD(m,n) = 1$, without loss of generality we may assume that m is odd implying that $T_m(\widetilde{a}) \neq T_m(\widetilde{b})$ for such $\widetilde{a}$ and $\widetilde{b}$ since $T_m(-1) = -1$, $T_m(1) = 1$. Similarly, $\beta^2 \neq 1$. Finally, observe that equalities (4.4) yield that $\alpha^{mn} = \pm 1$, $\beta^{mn} = \pm 1$, implying that

$$T_{mn}(\alpha) = \pm 1, \quad T_{mn}(\beta) = \pm 1. \tag{4.5}$$

In order to finish the proof observe that the equality $T_n(\cos\phi) = \cos n\phi$ implies easily that the polynomial T_n has exactly two critical values ± 1 and that the only points in the preimage $T_n^{-1}\{\pm 1\}$ which are not critical points

of T_n are the points ± 1. Therefore, the equalities (4.5) taking account that $\alpha \neq \pm 1$, $\beta \neq \pm 1$ imply that α and β are critical points of T_{mn} and hence critical points of P by the chain rule. $\square$

Theorem 4.2 combined with the second Ritt theorem allows us, at list in principle, to describe explicitely all the non-definite polynomials up to a given degree. In particular, the following statement holds:

Theorem 4.3 *For given $a \neq b$ non-definite polynomials $P \in \mathcal{P}_{11}$ appear only in degrees 6 and 10 and have, up to change $P \to \lambda \circ P$, where λ is a polynomial of degree one, the following form:*

1. $P_6 = T_6 \circ \tau$, where T_6 is the Chebyshev polynomial of degree 6 and τ is a polynomial of degree one transforming a, b into $-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}$.

2. $P_{10} = z^2 R^2(z^2) \circ \tau$, where $R(z) = z^2 + \gamma z + \delta$ is an arbitrary quadratic polynomial satisfying $R(1) = 0$ i.e. $\gamma + \delta = -1$, and τ is a polynomial of degree one transforming a, b into $-1, 1$.

Proof: First of all observe that if in Ritt's second theorem (Section 3.1 above) the degree of one of polynomials satisfying (3.4) is two, then solutions (3.6) may be written in form (3.5). Indeed, for odd n the equality

$$T_n(z) = zE_n(z^2) \tag{4.6}$$

holds for some polynomial E_n. Furthermore, $T_2 = \theta \circ z^2$, where $\theta = 2z - 1$, and hence

$$zE_n(z^2) \circ \theta \circ z^2 = T_n \circ T_2 = T_2 \circ T_n = \theta \circ T_n^2 = \theta \circ zE_n^2(z) \circ z^2.$$

Since the last equality implies the equality

$$zE_n(z^2) \circ \theta = \theta \circ zE_n^2(z),$$

we conclude that

$$T_n = \theta \circ zE_n^2(z) \circ \theta^{-1}, \quad T_{2n} = \theta \circ z^2 E_n^2(z^2). \tag{4.7}$$

Therefore, the equality

$$T_n \circ T_2 = T_2 \circ T_n$$

may be written in the form

$$(\theta \circ zE_n^2(z) \circ \theta^{-1}) \circ (\theta \circ z^2) = (\theta \circ z^2) \circ zE_n(z^2). \tag{4.8}$$

Now we are ready to prove the theorem.

Since each integer i, $2 \leq i < 11$, distinct from 6 or 10 is either a prime or a power of a prime, it follows from Corollary 4.1 that P is $[a,b]$-definite unless $\deg P = 6$ or $\deg P = 10$. It follows now from the second Ritt theorem and the remark above that if $\deg P = 10$, then P has the form given above. Similarly, if $\deg P = 6$, then $P = z^2 R^2(z^2) \circ \tau$, where R is a polynomial satisfying $R(1) = 0$. However, since in this case the degree of R equals one, up to change $P \to \lambda \circ P \circ \tau$, we obtain a unique polynomial $P = T_6$. $\square$

Let $V, V_1 \subset \mathcal{P}$ be fixed linear spaces. Let us denote by ND_{V,V_1} the set of polynomials $P \in V$ non-definite with respect to V_1. In particular, for $V = \mathcal{P}_d, V_1 = \mathcal{P}$ we denote the corresponding set by ND_d. If V_1 is a line spanned by a fixed $Q \in \mathcal{P}$ we write ND_{V,V_1} as $ND_{V,Q}$.

Proposition 4.1 *For each $V_1 \subset \mathcal{P}$ and $V \subset \mathcal{P}_d$ we have $ND_{V,V_1} \subset ND_d$. The dimension of ND_d does not exceed $[\frac{d}{6}] + 1$.*

Proof: The inclusion is immediate: any polynomial non-definite with respect to a smaller subspace is non-definite with respect to a larger one. By Theorem 4.2 the set ND_d consists of all $P \in \mathcal{P}_d$ which have $s \geq 2$ mutually $[a,b]$-prime right $[a,b]$-factors. Hence $ND_d \subset \cup_{s \geq 2} DEC_s^d(a,b)$. By Proposition 3.3 we have $\dim ND_d \leq [\frac{d}{6}] + 1$. This completes the proof. $\square$.

4.2 Co-definite polynomials

Let $[a,b]$ and a subspace $V \subset \mathcal{P}_{[a,b]}$ be given.

Theorem 4.4 *A polynomial Q is not V-co-definite if and only if there exists a polynomial $P \in V$ (necessarily non-definite) with a complete collection of $[a,b]$-indecomposable right factors $W_1, \dots, W_s$, $s \geq 2$, such that:*

1. *The polynomial Q can be represented as $Q = \sum_{j=1}^{s} S_j(W_j)$,*
2. *No one of $W_1, \dots, W_s$ is a right $[a,b]$-factor of Q.*

Proof: By Definition 4.1 a polynomial Q is not V-co-definite if and only if there exists a polynomial $P \in V$ such that all the moments $m_k = \int_a^b P^k(x) q(x) dx$, $k \geq 0$, vanish while P and Q do not satisfy the Composition Condition. Clearly, if such P exists it cannot be definite. Furthermore, by Theorem 4.1 the polynomial Q can be represented as a sum $Q = \sum_{j=1}^{s} S_j(W_j)$. Finally,

since P and Q do not satisfy Composition condition no one of $W_1,\dots,W_s$ can be an $[a,b]$-right factor of Q.

In the opposite direction, assume that $P \in V$ as required exists. Since Q possesses a representation $Q=\sum_{j=1}^{s} S_j(W_j)$, where $W_1,\dots,W_s$ are right $[a,b]$-factors of P, we conclude (by linearity of the moments in Q) that all the moments $m_k,\ k\geq 0$, vanish. Furthermore, since $W_1,\dots,W_s$ is a complete collection of right $[a,b]$-factors of P, the second assumption implies that P and Q do not satisfy the Composition Condition. Hence Q is not V-co-definite. $\square$

Definition 4.2 *For $V \subset \mathcal{P}$ we define the set $\mathcal{S}_{V,d} \subset \mathcal{P}_d$ as the set of polynomials $Q \in \mathcal{P}_d$ which can be represented as $Q=\sum_{j=1}^{s} S_j(W_j)$, where $W_1,\dots,W_s$ are all $[a,b]$-indecomposable right factors of a certain non-definite $P \in V$. The set $\mathcal{S}_V$ is the union $\cup_d \mathcal{S}_{V,d}$.*

By Theorem 4.4, in order to describe explicitly all V-co-definite polynomials up to degree d we have first to describe the set $\mathcal{S}_{V,d}$ and then to describe those $Q \in \mathcal{S}_{V,d}$ for which no one of $W_1,\dots,W_s$ is a right $[a,b]$-factor of Q. Both these questions in their general form turns out to be rather tricky, and we provide here only very partial results. To make formulas easier, without loss of generality we shall assume that $[a,b]$ coincides with $[-\frac{\sqrt{3}}{2},\frac{\sqrt{3}}{2}]$.

Theorem 4.5 *Let $V=\mathcal{P}_{9,[-\frac{\sqrt{3}}{2},\frac{\sqrt{3}}{2}]}$. Then the set $\mathcal{S}_{V,d}$ is a vector space consisting of all polynomials $Q \in \mathcal{P}_d$ representable as $Q=S_1(T_2)+S_2(T_3)$ for some polynomials $S_1,\ S_2$. Furthermore, the dimension $\mathcal{S}_{V,d}$ is equal to $[\frac{d+1}{2}]+[\frac{d+1}{3}]-[\frac{d+1}{6}]$. In particular, this dimension does not exceed $[\frac{2}{3}d]+1$.*

For $d\leq 4$ the space $\mathcal{S}_{V,d}$ coincides with $\mathcal{P}_d$, and starting with $d=5$ this space is always a proper subset of $\mathcal{P}_d$. We have $\mathcal{S}_{V,5}=\mathcal{P}_4\subset\mathcal{P}_5$ and $\mathcal{S}_{V,6}$ is the subspace in $\mathcal{P}_6$ consisting of all the polynomials Q of the form $Q=Q_1+\alpha T_3$ with Q_1 even of degree at most 6. $\mathcal{S}_{V,7}=\mathcal{S}_{V,6}$, while $\mathcal{S}_{V,8}$ is the subspace in $\mathcal{P}_8$ consisting of all the polynomials Q of the form $Q=Q_1+\alpha T_3$ with Q_1 even of degree at most 8.

Proof: By Theorem 4.3 the only non-definite polynomials in $V=\mathcal{P}_{9,[-\frac{\sqrt{3}}{2},\frac{\sqrt{3}}{2}]}$ are scalar multiples of T_6. $T_6=T_2\circ T_3=T_3\circ T_2$ has exactly two right $[-\frac{\sqrt{3}}{2},\frac{\sqrt{3}}{2}]$-factors T_2 and T_3. This proves the first claim of Theorem 4.5.

Next observe that
$$\mathbb{C}[T_n]\cap\mathbb{C}[T_m]=\mathbb{C}[T_l], \tag{4.9}$$

where $l = \mathrm{LCM}(n,m)$. Indeed if P is contained in $\mathbb{C}[T_n] \cap \mathbb{C}[T_m]$, then there exist polynomials A, B such that

$$P = A \circ T_n = B \circ T_m,$$

and in order to show that there exists a polynomial U such that $P = U \circ T_l$ one can use the second Ritt theorem. However, such a proof is more difficult that it seems since it requires an analysis of the possibility provided by (3.5) (see e.g. Lemma 4.1 of [18]). It is more convenient to observe that identity (4.3) implies that the function

$$F = P \circ \frac{1}{2}\left(z + \frac{1}{z}\right) = A \circ \frac{1}{2}\left(z^n + \frac{1}{z^n}\right) = B \circ \frac{1}{2}\left(z^m + \frac{1}{z^m}\right)$$

is invariant with respect to the both groups D_{2n} and D_{2m}, where D_{2s} is the dihedral group generated by the transformations $z \to 1/z$ and $z \to e^{2\pi i/s}z$. Therefore, F remains invariant with respect to the group $< D_{2n}, D_{2m} >= D_{2l}$ implying that that there exists a rational function U such that

$$F = U \circ \frac{1}{2}\left(z^l + \frac{1}{z^l}\right).$$

Since

$$U \circ \frac{1}{2}\left(z^l + \frac{1}{z^l}\right) = U \circ T_l \circ \frac{1}{2}\left(z + \frac{1}{z}\right),$$

we conclude that $P = U \circ T_l$, and it is easy to see that U actually is a polynomial.

Denote by $U_{d,n}$ the subspace of $\mathbb{C}[T_n]$ consisting of all polynomials of degree $\leq d$. By the remark above we have $U_{d,n} \cap U_{d,m} = U_{d,l}$. This implies that

$$\dim \mathcal{S}_{V,d} = \dim U_{d,2} \oplus U_{d,3} - 2 = \dim U_{d,2} + \dim U_{d,3} - \dim U_{d,6} - 2 =$$

$$= [\frac{d+1}{2}] + [\frac{d+1}{3}] - [\frac{d+1}{6}] - 1 \leq [\frac{2}{3}d] + 1.$$

A description of $\mathcal{S}_{V,d}$ for $d \leq 8$ is obtained by a straightforward computation. This completes the proof of Theorem 4.5. $\Box$

Theorem 4.6 *A polynomial of the form $Q = S_1(T_2) + S_2(T_3)$, where S_1, S_2 are non-zero polynomials, has T_2 (resp. T_3) as its right factor if and only if S_2 is a polynomial in T_2 (resp. S_1 is a polynomial in T_3).*

Proof: Indeed, assume say that $S_1(T_2)+S_2(T_3)=R(T_2)$ for some polynomial R. Then by (4.9) there exists a polynomial F such that

$$S_2\circ T_3=F\circ T_6=F\circ T_2\circ T_3$$

implying that $S_2=F\circ T_2$. $\square$

Corollary 4.3 *Let $V=\mathcal{P}_{9,[-\frac{\sqrt{3}}{2},\frac{\sqrt{3}}{2}]}$. A polynomial $Q\in\mathcal{P}_{8,[-\frac{\sqrt{3}}{2},\frac{\sqrt{3}}{2}]}$ is not V-co-definite if and only if it can be represented in the form*

$$Q=R+\alpha T_3,\quad \alpha\in\mathbb{C},\tag{4.10}$$

where $\alpha\neq 0$, and $R\in\mathcal{P}_{8,[-\frac{\sqrt{3}}{2},\frac{\sqrt{3}}{2}]}$ is an even polynomial distinct from $\beta T_6+\gamma$, $\beta,\gamma\in\mathbb{C}$.

Proof: By the above results, if $P\in\mathcal{P}_8$ is not co-definite it can be represented in the form $Q=S_1(T_2)+S_2(T_3)$, where $\deg S_1\leq 4$, and S_1 is not a linear polynomial in T_3, while $\deg S_2\leq 2$, and S_2 is not a linear polynomial in T_2. Since S_2 can be represented in the form $\delta T_2+\alpha z+\kappa$, where $\delta,\alpha,\kappa\in\mathbb{C}$, we conclude that such Q can be represented in the form

$$Q=\widetilde{S}_1(T_2)+\alpha T_3,\tag{4.11}$$

where $\deg\widetilde{S}_1\leq 4$. Furthermore, $\alpha\neq 0$, since otherwise Q is a polynomial in T_2, and $\widetilde{S}_1$ is not a linear polynomial in T_3, since otherwise Q is a polynomial in T_3. Therefore, since $\mathbb{C}[T_2]=\mathbb{C}[z^2]$ and $T_3\in\mathcal{P}_8$, the polynomial P admits representation (4.10).

In other direction, it follows from (4.10) that (4.11) holds, where $\alpha\neq 0$ and $\widetilde{S}_1\neq\beta T_3+\gamma$, $\beta,\gamma\in\mathbb{C}$, implying that Q is not co-definite. $\square$

4.3 Polynomials with a special structure

Let $\mathcal{R}=\{r_1,r_2,\dots\}$ be a set of prime numbers, finite or infinite. Define $U(\mathcal{R})$ as a subspace of $\mathcal{P}$ consisting of polynomials $P=\sum_{i=0}^N a_ix^i$ such that for any non-zero coefficient a_i the degree i is either coprime with each $r_j\in\mathcal{R}$ or is a power of some $r_j\in\mathcal{R}$. Similarly, define $U_1(\mathcal{R})$ as a subspace of $\mathcal{P}$ consisting of polynomials P such that for any non-zero coefficient a_i of P all prime factors of i are contained in $\mathcal{R}$. In particular, if $\mathcal{R}$ coincides with the set of all primes numbers, then $U(\mathcal{R})$ consists of polynomials in $\mathcal{P}$ whose degrees with non-zero coefficients are powers of primes, while $U_1(\mathcal{R})=\mathcal{P}$.

Theorem 4.7 *Let $\mathcal{R} = \{r_1, r_2, \dots\}$ be fixed. Then for any $a \neq b$ each polynomial $P \in U(\mathcal{R})$ is $[a,b]$-definite, and, in particular, it is $[a,b]$-definite with respect to $U_1(\mathcal{R})$, and each $Q \in U_1(\mathcal{R})$ is $[a,b]$-co-definite with respect to $U(\mathcal{R})$.*

Proof: We show that vanishing of all the moments $m_k = \int_a^b P^k(x)q(x)dx$ for $P \in U(\mathcal{R})$ and $Q \in U_1(\mathcal{R})$ implies Composition Condition. By the construction, the degree of any $Q \in U_1(\mathcal{R})$ is the product of certain prime numbers in $\mathcal{R}$. By Corollary 4.3 of [16] vanishing of the moments implies that the degrees of P and Q cannot be mutually prime. Hence $\deg P$ is divisible by one of r_j. But then by the construction this degree must be a power of r_j. Finally, it was shown in [16] (see also Section 3.2.1 above) that polynomials P with $\deg P$ a power of a prime number are definite. Hence vanishing of the moments m_k implies Composition condition for P, Q on $[a,b]$. $\square$

4.4 The Moment and the Composition Sets

Using the information on definite and co-definite polynomials provided above we now can describe more accurately the interrelation between the Moment and the Composition sets.

Let $V, V_1 \subset \mathcal{P}$ be fixed linear spaces. As above, ND_{V,V_1} is the set of polynomials $P \in V$ non-definite with respect to V_1.

Theorem 4.8 *For each $Q \in V_1$ we have $MS_{V,Q} = COS_{V,Q} \cup N$ where N is contained in $ND_{V,V_1} \subset ND$. In particular, for $V \subset \mathcal{P}_d$ and any Q the dimension of N is at most $[\frac{d}{6}] + 1$.*

Proof: If $P \in MS_{V,Q}$ but P is not in $COS_{V,Q}$ then P is not definite with respect to V_1, and hence it belongs to ND_{V,V_1}, which is always a subset of ND. If $V \subset \mathcal{P}_d$ then $P \in ND_d$ and the bound on the dimension follows from Proposition 4.1. $\square$

Example ([3]) Let $[a,b] = [-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}]$. Put $Q = (T_2 + T_3)$, and consider $V = \mathcal{P}_6$. Then the Moment set $MS_{V,Q}$ contains exactly two components: the composition component $COS_{V,Q} = \{P = R(T_2 + T_3)\}$, with R any polynomial of degree 2, and the non-composition component $\mathcal{T} = \{P = \alpha T_6,\ \alpha \in \mathbb{C}\}$. Here $\mathcal{T}$, in fact, coincides with $ND_{V,Q}$.

Our description of co-definite polynomials in Section 4.2 produces the following result on the Moment and Composition Sets:

Corollary 4.4 *Let $V \subset \mathcal{P}$, and let $V_1 \subset \mathcal{P}_d$ be such that $V_1 \cap \mathcal{S}_{V,d} = \{0\}$, in the notation of Definition 4.2. Then for each $Q \in V_1$ we have $ND_{V,V_1} = \emptyset$ and $MS_{V,Q} = COS_{V,Q}$.*

Proof: By Theorems 4.4 and via Definition 4.2, each $Q \in V_1,\ Q \neq 0$ is co-definite with respect to V. Consequently, each $P \in V$ is definite with respect to such Q. Application of Theorem 4.8 completes the proof. □

In situation of Section 4.3 we get

Corollary 4.5 *For a fixed set $\mathcal{R}$ of prime numbers let $V = U(\mathcal{R}),\ V_1 = U_1(\mathcal{R})$, in notations of Section 4.3. Then for each $Q \in V_1$ we have $MS_{V,Q} = COS_{V,Q}$.*

Proof: The result follows directly from Theorems 4.8 and 4.7. □

5 Center Set near infinity

Let a polynomial Q and a linear subspace $V \subset \mathcal{P}_d$ be fixed. In this section we analyze the structure of the Center Set $CS_{V,Q}$ at and near the infinite hyperplane HV, as compared to the Moment and Composition Sets $MS_{V,Q}$ and $COS_{V,Q}$. By Proposition 2.1 we have at infinity $\bar{COS} \subset \bar{CS} \subset \bar{MS}$.

An important fact is that for each *definite* $P_0 \in \bar{CS}$ there is an entire projective neighborhood U of P_0 in PV where CS and COS coincide:

Theorem 5.1 *Let $P_0 \in \bar{CS}_{V,Q}$ be a definite polynomial. Then*

1. In fact, $P_0 \in \bar{COS}_{V,Q}$.

2. There exists a projective neighborhood U of P_0 in PV such that $CS_{V,Q} \cap U = COS_{V,Q} \cap U$.

3. $CS_{V,Q} \cap U$ is a linear space defined by vanishing of the linear parts of the Center Equations. In particular, CS is regular in U and its local ideal is generated by the Center Equations.

Proof: From the inclusion $\bar{CS} \subset \bar{MS}$ we get $P_0 \in \bar{MS}_{V,Q}$. Since the polynomial P_0 is definite by the assumptions, moments vanishing for this polynomial implies composition, so in fact $P_0 \in \bar{COS}_{V,Q}$.

In homogeneous coordinates (P, ν) in PV near P_0 put $P = P_0 + P_1,\ P_1 \in V$. By Proposition 7.2 of [3] the only nonzero linear terms in the expansions

of the homogenized Center Equations around the point $(0,0)$ in variables $P_1,\ \nu$ are given by the following linear functionals in P_1:

$$L_k(P_1) = -(k-3)\int_a^b P_0^{k-4}(x)q(x)P_1(x)dx,\ k = 4,5,\dots. \quad (5.1)$$

Denote by $L \subset V$ the subspace defined by the linear equations $L_k(P_1) = 0, \quad k = 4,5,\dots.$ Let us show first that $L \subset COS_{V,Q}$. Consider certain polynomial $P_1 \in L$. Since P_0 is definite, vanishing of $L_k(P_1)$ implies composition condition for $P_0(x)$ and $S(x) = \int_a^x P_1(\tau)q(\tau)d\tau$. Since, being definite, P_0 has only one $[a,b]$-prime right composition $[a,b]$-factor W, we conclude that $S = \widetilde{S}(W)$. By the same reason, from $P_0 \in COS_{V,Q}$ it follows that $Q = \widetilde{P}(W)$. Now Lemma 7.3 in [3] implies that $P_1 = \widetilde{P_1}(W)$, i.e. $P_1 \in COS_{V,Q}$, and hence $L \subset COS_{V,Q}$.

It follows that all the Center Equations vanish on L, which is the zero set of their linear parts. Now we are in a situation of Lemma 7.4 of [3] (Nakayama Lemma in Commutative algebra - see for example [10], chapter 4, lemma 3.4). The conclusion is that $CS = L = COS$ in a neighborhood of P_0, and the local ideal of this set is generated by the Center Equations. This completes the proof of Theorem 5.1. □

6 Main results

Let $a \neq b$ be fixed. Below we denote by $\widetilde{T}_j$ the transformed Cebyshev polynomials $\widetilde{T}_j = T_j \circ \mu$, μ being a linear polynomial transforming the couple (a,b) to the couple $(-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2})$.

Let linear subspaces $V, V_1 \subset \mathcal{P}_{[a,b]}$ and a polynomial $Q \in V_1$ be fixed. The affine Center Set $CS_{V,Q}$ always contains the Composition Set $COS_{V,Q}$. In this section we provide an upper bound for the dimensions of affine *non-composition* components in CS. As above, $ND_{V,V_1} \subset ND$ denotes the set of V_1 non-definite polynomials in V. For each affine algebraic set $A \subset V$ let $\bar{A}$ denote the intersection of A with the infinite hyperplane HV.

Theorem 6.1 *For each irreducible non-composition component A of the affine Central Set $CS_{V,Q}$ we have $\bar{A} \subset \bar{CS}_{V,Q} \cap ND \subset \bar{M}S_{V,Q} \cap ND$. Consequently,* $\dim A \leq \dim(\bar{M}S_{V,Q} \cap ND) + 1$. *In particular, for any polynomial Q, and $V \subset \mathcal{P}_d$ the dimension of A cannot exceed $[\frac{d}{6}] + 2$.*

Proof: We always have $\bar{A} \subset \bar{C}S \subset \bar{M}S$. Now, if $P_0 \in \bar{A}$ then P_0 cannot be definite. Indeed, otherwise there would exist a neighborhood U of P_0 provided by Theorem 5.1, where $A \cap U \subset COS \cap U$. Since A is irreducible, this would imply that $A \subset COS$, which contradicts the assumption that A is a non-composition component of CS. Thus $\bar{A} \subset \bar{M}S_{V,p} \cap ND$. Now since the infinite hyperplane HV has codimension one in the projective space PV, for each A we have $\dim A \leq \dim \bar{A} + 1$. Application of Proposition 4.1 completes the proof. $\square$.

Notice that the dimension of the composition components of CS may be of order $\frac{d}{2}$, while by Theorem 6.1 the dimension of the non-composition components is of order at most $\frac{d}{6.}$ To our best knowledge, this is the first general bound of this form for the polynomial Abel equation.

Corollary 6.1 *([3]). Let $V = \mathcal{P}_5$. Then for any Q the Center Set $CS_{V,Q}$ consists of a Composition Set with possibly a finite number of additional points.*

Proof: By Theorem 4.3 there are no non-definite polynomials in $V = \mathcal{P}_5$. So the set $\bar{M}S_{V,Q} \cap ND$ is empty and its dimension is -1. $\square$.

Corollary 6.2 *Let $V = \mathcal{P}_9$. Then for any Q the Center Set $CS_{V,Q}$ consists of a Composition Set with possibly a finite number of additional curves.*

Proof: By Theorem 4.3 the only non-definite polynomials in $V = \mathcal{P}_9$ are scalar multiples of $\widetilde{T}_6$. So the set $\bar{M}S_{V,Q} \cap ND$ consists at most of one point, and its dimension is at most 0. $\square$

Corollary 6.3 *Let $V = \mathcal{P}_{11}$. Then for any Q the $CS_{V,Q}$ consists of a Composition Set with possibly a finite number of additional two-dimensional components.*

Proof: Theorem 4.3 describes non-definite polynomials in $V = \mathcal{P}_{11}$. We see that the set $\bar{M}S_{V,Q} \cap ND$ consists at most of a finite number of points, and a one-dimensional component, and its dimension is at most 1. $\square$.

Notice that the bounds of Corollaries 6.1-6.3 are more accurate than the general bound of Theorem 4.3.

Recall that by Definition 4.2 the set $\mathcal{S}_V$ consists of all Q which can be represented as $Q = \sum_{j=1}^{s} S_j(W_j)$, where $W_1, \ldots, W_s$ are all $[a,b]$-indecomposable right factors of a certain $P \in V$.

Theorem 6.2 *Let $V \subset \mathcal{P}$, and let $Q \in \mathcal{P} \setminus \mathcal{S}_V$. Then the Center Set $CS_{V,Q}$ consists of a Composition Set with possibly a finite number of additional points. In particular, this is true for $V = \mathcal{P}_9$ and any Q not representable as $Q = S_1(\widetilde{T}_2) + S_2(\widetilde{T}_3)$.*

Proof: This result follows directly from Theorem 4.3 and Corollary 4.4. The case $V = \mathcal{P}_9$ is covered by Theorem 4.5. However, since Theorem 6.2 is one of the central results of this paper, we give its short independent proof. We show that the Moment Set $MS_{V,Q}$ does not contain non-definite polynomials. Indeed, for each non-definite $P \in V$ vanishing of the moments $m_k = \int_a^b P^k(x)q(x)dx$ implies $Q \in \mathcal{S}_V$, by Definition 4.2. But by our assumptions $Q \in \mathcal{P} \setminus \mathcal{S}_V$. Therefore P is not in $MS_{V,Q}$. Application of Theorem 6.1 completes the proof. $\square$.

We expect that the result of Theorem 6.2 can be extended as follows:

Conjecture 2 *Let $V \subset \mathcal{P}$. Assume that either $Q \in \mathcal{P} \setminus \mathcal{S}_V$, or $Q \in \mathcal{S}_V$, and it is not V-co-definite. Then the Center Set $CS_{V,Q}$ consists of a Composition Set with possibly a finite number of additional points.*

Closely related to Conjecture 2 is the following

Conjecture 3 *For polynomials P, Q vanishing of all the moments $m_k(P,Q)$ and of all the second Melnikov coefficients $D_j(P,Q)$ (see Theorem 2.1) implies Composition Condition.*

Theorem 6.3 *Conjecture 3 implies Conjecture 2.*

Proof: Assume, as in Conjecture 2, that either $Q \in \mathcal{P} \setminus \mathcal{S}_V$, or $Q \in \mathcal{S}_V$, and it is not V-co-definite. The first case is treated in Theorem 6.2. In the second case we still show that the Center Set at infinity $\bar{CS}_{V,Q}$ does not contain non-definite polynomials. Assume, in contradiction, that $P \in \bar{CS}_{V,Q}$ is non-definite, and let $W_1, \ldots, W_s,\ s \geq 2$, be all the $[a,b]$-indecomposable right factors of P. According to Theorem 2.1 P satisfies equations $m_k(P,Q) = 0$ and $D_j(P,Q) = 0$. By the first set of these equations $Q = \sum_{j=1}^{s} S_j(W_j)$, and by the second set and by Conjecture 3 we conclude that one of W_j is a right factor of Q. Now according to Theorem 4.4 Q is V-co-definite, in contradiction with the assumptions. This completes the proof. $\square$

Our next result confirms Conjectures 2 and 3 for $\deg P \leq 9,\ \deg Q \leq 8$.

Theorem 6.4 *Let* $\deg P \leq 9,\ \deg Q \leq 8$. *Then vanishing of all the moments* $m_k(P,Q)$ *and of three initial second Melnikov coefficients* $D_j(P,Q)$ *implies Composition Condition.*

In particular, for $V = \mathcal{P}_9$, *and for any* Q *of degree up to* $\underset{\sim}{8}$ *not of the form* $Q = S_1(\widetilde{T}_2) + S_2(\widetilde{T}_3)$, *or of this form, but such that neither* $\widetilde{T}_2$ *nor* $\widetilde{T}_3$ *are the right composition factors of* Q, *the center set* $CS_{V,Q}$ *consists of a composition set with possibly a finite number of additional points.*

Proof: Let polynomials P,Q with $\deg P \leq 9,\ \deg Q \leq 8$ be given. If $P \neq \alpha\widetilde{T}_6$, it is definite, and hence vanishing of all the moments $m_k(P,Q)$ implies Composition Condition for P,Q. Consider now the case $P = \widetilde{T}_6$. Here vanishing of $m_k(P,Q)$ implies that Q has a form $Q = S_1(\widetilde{T}_2) + S_2(\widetilde{T}_3)$ for some polynomials S_2 and S_3. By Theorem 4.5 we conclude that Q can be written as $Q = S_1(\widetilde{T}_2) + \alpha\widetilde{T}_3$, with $S_1(T) = \sum_{i=1}^4 c_i T^i$. Now we use the second set of equations $D_j(P,Q) = 0$.

Proposition 6.1 *The first four equations at infinity* $D_j(P,Q) = 0$ *given in Theorem 2.1 can be written as*

$$D_1(P,Q) = \int_a^b Q^2 p = 0, \qquad D_2(P,Q) = \int_a^b Q^2 P p = 0,$$

$$D_3(P,Q) = 2\int_a^b Q^2 P^2 p + \int_a^b Q(t)P(t)p(t)dt \int_a^t Q(\tau)p(\tau)d\tau = 0.$$

$$D_4(P,Q) = 2\int_a^b Q^2 P^3 p + \int_a^b Q(t)P^2(t)p(t)dt \int_a^t Q(\tau)p(\tau)d\tau = 0.$$

Proof: Straightforward, but rather lengthy computation. $\square$

The following results describe the application of these three equations to the specific combinations of Chebyshev polynomials representing Q. To simplify the numeric coefficients we assume here that $[a,b] = [0,1]$ and so $\widetilde{T}_2(x) = x(x-1), \widetilde{T}_3(x) = x(x-1)(2x-1)$. We also have $\widetilde{T}_6 = \widetilde{T}_3^2 = \widetilde{T}_2^2 + 4\widetilde{T}_2^3$.

Proposition 6.2 *Let* $P = \widetilde{T}_6, Q = S_1(\widetilde{T}_2) + \alpha\widetilde{T}_3$, *with* $S_1(T) = \sum_{i=1}^3 c_i T^i$. *If the first two equations of Proposition 6.1 are satisfied, then either* $Q = S_1(\widetilde{T}_2)$ *or* $Q = \beta\widetilde{T}_6 + \alpha\widetilde{T}_3$. *In each of these cases* Q *has either* $\widetilde{T}_2$ *or* $\widetilde{T}_3$ *as a right composition factor.*

Proof: Substitution to the first two equations of Proposition 6.1 gives the following system of equations on the coefficients α, c_1, c_2, c_3:

$$\alpha(-13c_1 + 4c_2 - c_3) = 0, \ \alpha(-\frac{38}{3}c_1 + 4c_2 - c_3) = 0.$$

The result follows immediately from this system. $\square$

Proposition 6.3 *Let* $P = \widetilde{T}_6, Q = S_1(\widetilde{T}_2) + \alpha \widetilde{T}_3$*, with* $S_1(T) = \sum_{i=1}^{4} c_i T^i$*. If all the three equations of Proposition 6.1 are satisfied, then either* $Q = S_1(\widetilde{T}_2)$ *or* $Q = \beta \widetilde{T}_6 + \alpha \widetilde{T}_3$*. In each of these cases* Q *has either* $\widetilde{T}_2$ *or* $\widetilde{T}_3$ *as a right composition factor.*

Proof: Substitution to the equations of Proposition 6.1 gives the following system of equations on the coefficients $\alpha, c_1, c_2, c_3, c_4$:

$$\alpha(-13c_1 + 4c_2 - c_3 + \frac{4}{17}c_4) = 0, \ \alpha(-\frac{38}{3}c_1 + 4c_2 - c_3 + \frac{16}{69}c_4) = 0,$$

$$\alpha(-\frac{325}{26}c_1 + 4c_2 - c_3 + \frac{20}{87}c_4) = 0.$$

The result follows immediately from this system. $\square$

Combining Propositions 6.2 and 6.3 we complete the proof of Theorem 6.4: vanishing of the moments and of the initial three Melnikov coefficients implies composition. $\square$

Finally we consider Center Sets in the subspaces $V = U_{\mathcal{R}}$, as defined in Section 3.3.

Theorem 6.5 *Let a subset* $\mathcal{R} = \{r_1, r_2, \dots\}$ *of prime numbers be fixed. Put* $V = U(\mathcal{R})$*, as defined in Section 3.3 above. Then for any* $a \neq b$ *and for each fixed polynomial* $Q \in U_1(\mathcal{R})$ *the center set* $CS_{V,Q}$ *of Abel equation (1.1) inside the space* V *consists of a Composition Set with possibly a finite number of additional points.*

Proof: This is a direct consequence of Corollary 4.5 and Theorem 6.1. $\square$

The results of this section cover all the results of Theorems 1.1 - 1.4 stated in the Introduction.

The methods developed in this paper work not only in the setting of the Center Equations at infinity. They can be applied also to the study of the

local structure of the affine Center Set, and to the parametric versions of the Center-Focus problem for Abel equation (see [6, 7, 9] for recent developments in this direction). We plan to present these results separately.

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